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# T-DUALITY WITHOUT QUANTUM CORRECTIONS

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## ABSTRACT

It is a well known fact that the classical (“Buscher”) transformations of T-duality do receive, in general, quantum corrections. It is interesting to check whether classical T-duality can be exact as a quantum symmetry. The natural starting point is a  $\sigma$ -model with  $N=4$  world sheet supersymmetry. Remarkably, we find that (owing to the fact that  $N=4$  models with torsion are *not* off-shell finite as quantum theories), the T-duality transformations for these models get in general quantum corrections, with the only known exception of warped products of flat submanifolds or orbifolds thereof with other geometries.

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# 1 Introduction

There has recently been some interest on quantum corrections to T-duality transformations [3] (where by “classical” T-duality we will understand in all this work Buscher’s transformations [1] (not including the dilaton transformation). In most works, however, the problem has been set up as a pure quantum field theoretical one, forgetting conformal invariance, which is one of the major themes in T-duality as a string symmetry, and, indeed, the origin of the dilaton transformation [4].

Our point of view in this work has been, in a sense, the opposite one; that is, start from a conformal invariant theory as well behaved quantum-mechanically as possible, and examine whether in this simplest of all interesting contexts, T-duality can be implemented *without* quantum corrections, perhaps as a Legendre transform [5].

General sigma-models with N=4 (world-sheet) supersymmetry are widely believed to be finite as quantum theories [14]. ( Ricci flat N=2, on the other hand, is not enough, because of the well-known counterterms of Grisaru, van de Ven and Zanon [7]). This set of models is then the most promising starting point for our purposes.

Although it is expected on general grounds that the dual model is physically equivalent to the original one, it is by now quite clear that N=4 supersymmetry will not always be manifest in the dual model [10]. One can classify all isometries in two types, translational and rotational, whose essential distinction is whether or not they have fixed points. One can be slightly more precise in four dimensional target spaces, where a translational isometry is by definition such that the (antisymmetric part of) the covariant derivative of the Killing vector is (anti) self-dual:

$$\nabla_\mu k_\nu = \pm \frac{1}{2} \sqrt{|g|} \epsilon_{\mu\nu}^{\rho\sigma} \nabla_\rho k_\sigma \quad (1)$$

When the isometry possesses fixed points (which appear as zeros of the Killing vector field), supersymmetry can still be preserved if the rank of the matrix  $M_{\mu\nu} \equiv \nabla_\mu k_\nu$  at the fixed point is equal to 4 (nut), and the isometry is translational. On the other hand, when the corresponding rank is 2 (bolt), manifest supersymmetry is lost.

But even if the problem is restricted to translational isometries (without fixed points), where N=4 supersymmetry is expected to be manifest in the dual model, a potential paradox immediately comes to mind. The general framework of N=4 does not force  $g_{00} = 1$  in adapted coordinates to the Killing  $k = \partial_0$ . Buscher’s formulas then imply a dilaton in the dual model  $\phi = -\frac{1}{2} \log g_{00}$ , spoiling at least the off-shell finiteness of the dual theory.

If the gauging procedure [13] is used, one can regulate the determinant coming from the integration of the gauge fields in such a way that the correct dilaton is obtained [1].

The auxiliar gauged theory corresponding to a general (1,1) model, reads:

$$S_{auxiliar} = -\frac{1}{2\pi} \int d^2z d^2\theta ((g_{\mu\nu} + B_{\mu\nu}) \nabla_+ X^\mu \nabla_- X^\nu - \tilde{X}(D_+ V_- + D_- V_+)) \quad (2)$$

where the notation of reference [9] has been used, together with  $\nabla_\pm X^\mu \equiv D_\pm X^\mu + k^\mu A_\pm$ ,  $A_\pm$  being the gauge superfields,  $k^\mu$  the Killing vector, and  $\tilde{X}$  the Lagrange supermultiplier.

Performing now the (gaussian) integral over the gauge superfields the standard dual path

integral is obtained up to the term :

$$\int DA_+ DA_- \exp\left(-\frac{1}{2} \int d^2z d^2\theta g_{00}(X^i) A_+ A_-\right) \quad (3)$$

Writing the above path integral in components and integrating the part corresponding to the gauge supermultiplet we get two determinants (one of them coming from the bosonic components and the other from the fermionic ones) which naively cancel.

We know, however, that we can not trust this formal argument, because of the fact that the beta functions of both the metric and the antisymmetric tensor remain the same as in the purely bosonic model, for which a dilaton transformation is necessary.

In order to possess N=4 supersymmetry, in models without torsion the target space has to be endowed with a hyperkähler structure. [14]. It is rather clear that if we want the dual model to have also manifest N=4 supersymmetry, the Lie derivative of the complex structures with respect to the Killing must be zero.

$$\mathcal{L}_k J^{(X)} = 0 \quad (4)$$

This implies that in adapted coordinates the complex structures are independent of the cyclic coordinates. When this condition is satisfied, the isometry is said to be triholomorphic. In 4-dimensional target spaces triholomorphicity is equivalent to (anti)self-duality [15].

Although we are interested in target spaces of arbitrary dimensions, we shall make our considerations explicit for dimension 4, because in this case the general form of the metric tensor for hyperkähler manifolds with one translational Killing isometry is [15]

$$ds^2 = V(d\tau + w_i dx^i)^2 + V^{-1}(dx^i dx^i) \quad (5)$$

with the conditions

$$\partial_{[j} w_{k]} = \frac{(\pm 1)}{2} \epsilon_{jki} \partial_i V^{-1} \quad (6)$$

in adapted coordinates to the Killing vector  $\frac{\partial}{\partial \tau}$ .

General N=4 models are widely believed to be finite to all orders in perturbation theory [18], but only when the torsion vanishes they are off-shell finite as well( [14][6] ). In order to stay in as firm a ground as possible, we shall then restrict our attention to this subset of finite models.

## 2 Translational duality in general four-dimensional N=4 models

Buscher's formulas for the dual background yield:

$$d\tilde{s}^2 = V^{-1}(d\tilde{\tau}^2 + d\tilde{x}^i d\tilde{x}^i) \quad (7)$$

$$\tilde{b} = 2w_i d\tilde{\tau} \wedge d\tilde{x}^i \quad (8)$$

It has all N=4 supersymmetries manifest because we have dualized with respect to a triholomorphic isometry [16, 17]. Standard wisdom would then suggest [18] it to be finite as well. To be specific, the dual left-right Kähler forms are

$$\tilde{J}_-^i = V^{-1}(\pm d\tilde{\tau} \wedge d\tilde{x}^i - \frac{1}{2}\epsilon^{ijk}d\tilde{x}^j \wedge d\tilde{x}^k) \quad (9)$$

$$\tilde{J}_+^i = V^{-1}(\mp d\tilde{\tau} \wedge d\tilde{x}^i - \frac{1}{2}\epsilon^{ijk}d\tilde{x}^j \wedge d\tilde{x}^k) \quad (10)$$

A straightforward computation of the beta function corresponding to the metric tensor [20, 21] gives, however, a non-zero result, namely:

$$\beta_{\mu\nu} = \tilde{R}_{\mu\nu}^+ = -2\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\Phi} \neq 0 \quad (11)$$

where  $\tilde{R}_{\mu\nu\sigma\rho}^+$  is the Riemann curvature for the connection

$$\Gamma_{\rho\sigma}^\mu \equiv \{ {}_{\rho\sigma}^\mu \} + T_{\rho\sigma}^\mu; \quad (12)$$

the torsion is defined through  $\tilde{T}_{\mu\nu\sigma} \equiv \frac{1}{2}(\partial_\mu\tilde{b}_{\nu\sigma} + \partial_\sigma\tilde{b}_{\mu\nu} + \partial_\nu\tilde{b}_{\sigma\mu})$ , and the dilaton is given in terms of the metric function by  $\tilde{\Phi} \equiv \frac{1}{2}\log V$ .

According to standard wisdom ([6]), this means that the dual model is indeed finite, but only on-shell; that is, after a field redefinition.

The fact that a dilaton is needed in the dual model in order for it to be one-loop conformally invariant means, of course, that the (manifestly) N=4 world-sheet supersymmetric model given by (7) is not conformally invariant by itself.

It is not known, however, whether the preceding one-loop dilaton correction is enough to ensure conformal invariance to all orders.

### 3 Duality under translational isometries in N=4 models with metric of the warped product type

In this section, the sigma model will be assumed to have n commuting isometries,  $k_a$ . In adapted coordinates,  $k_a \equiv \frac{\partial}{\partial y^a}$ , the most general sigma model reads:

$$S = \frac{1}{2\pi} \int d^2z d^2\theta E_{ab}(X) D_+ Y^a D_- Y^b + g_{ij}(X) D_+ X^i D_- X^j + F(X)_{ai} D_+ Y^a D_- X^i + F(X)_{ia} D_+ X^i D_- Y^a \quad (13)$$

(denoting by  $x^i$ ,  $i = n+1, \dots, d$ , the coordinates not adapted to the isometries.)

It has been shown in the preceding section that even in the restricted N=4 framework, models with torsion are not always finite. The original model will be torsion free when  $B_{ab} \equiv E_{[ab]} = F_{[ai]} = 0$ , and the dual model when the “mixed” terms  $F_{ai} = F_{ia} = 0$ ; we further define the matrix  $G \equiv G_{ab} \equiv E_{(ab)}$ .

The Ricci tensor for the above models reads, under those conditions:

$$R_{ab} = \frac{1}{2}g^{ij}(G \square_{ij} G + \frac{1}{2}\partial_k GG^{-1}\partial^k G)_{ab} \quad (14)$$

$$R_{ai} = 0 \quad (15)$$

$$R_{ij} = \hat{R}_{ij} + \frac{1}{2} \text{tr}(\square_{ij}G + \frac{1}{2}G^{-1}\partial_iGG^{-1}\partial_jG) \quad (16)$$

where a caret over an object means that it has been computed with the quotient metric,  $G_{ij}(x)$ , and the operator  $\square_{ij}$  is defined by

$$(G\square_{ij}G)_{ab} \equiv \hat{\nabla}_i\hat{\nabla}_jG_{ab} - \frac{1}{2}(\partial_iGG^{-1}\partial_jG + \partial_jGG^{-1}\partial_iG)_{ab} \quad (17)$$

In the simple setting considered in this section, in which there is no torsion neither in the initial, nor in the dual model, the dual metric is obtained by the transformation  $G \rightarrow G^{-1}$  (this being actually the reason for introducing the convenient ‘‘covariant’’ operator  $\square_{ij}$ , transforming as  $\square_{ij}G^{-1} = -G(\square_{ij}G)G^{-1}$ ).

Ricci flatness of both the original and the dual model (a necessary condition for them to be hyperkähler) leads to

$$\text{tr } G^{-1}\partial_iG = 0 \quad (18)$$

which means that the corresponding dilaton  $\det G = \text{const.}$

Conformal invariance then reduces to the conditions

$$(g^{ij}\square_{ij}G)_{ab} = 0 \quad (19)$$

$$\hat{R}_{ij} = -\frac{1}{4}\text{tr}G^{-1}\partial_iGG^{-1}\partial_jG \quad (20)$$

As a particular instance of the above, four-dimensional hyperkähler target spaces admit a further triholomorphic Killing vector provided the metric can be written in coordinates adapted to the Killings,  $\frac{\partial}{\partial\tau}$  and  $\frac{\partial}{\partial z}$  as:

$$\begin{aligned} ds^2 &= V(x,y)(d\tau)^2 + 2\omega(x,y)V(x,y)d\tau dz + \\ &(V(x,y)\omega(x,y)^2 + V^{-1}(x,y))dz^2 + V^{-1}(x,y)(dx^2 + dy^2) \end{aligned} \quad (21)$$

and

$$\partial_xV^{-1} = \pm\partial_y\omega \quad (22)$$

$$\partial_yV^{-1} = \mp\partial_x\omega \quad (23)$$

It is curious to notice that the above equations can be interpreted as the Cauchy-Riemann conditions for the analytic function  $f(x \pm iy) \equiv V^{-1} \pm i\omega$ . On the other hand, it is not difficult to show that only flat manifolds admit further holomorphic isometries.

In our previous notation,  $\det G = 1$ , and moreover  $G^{-1} = G$  up to relabellings of the coordinates, conveying the fact that the model is geometrically T-self-dual (up to boundary conditions; this includes the usual  $R \rightarrow \frac{1}{R}$  transformations in the toroidal case).

## 4 Conclusions

We have obtained the rather surprising result that T-duality does not act “classically”,(i.e., without quantum corrections), even in the simplest of all sigma-models, namely those which enjoy N=4 world-sheet supersymmetry.

The reason for that stems from the fact that (as we have explicitly checked), N=4 models with torsion are not always conformally invariant to all loops.

The only physical situation we have found in which T-duality acts as a *classical* symmetry is the one in which the metric correspond to a warped product. This includes as particular examples, (warped) products of tori (orbifolds) with other manifolds.

Actually, even in four dimensional target spacetimes, we have not been able to prove that those are the *only* allowed classical situations. The further study of the general class of N=4 models with torsion in arbitrary target space dimension is obviously one of the outstanding open problems in this context.

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